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On the Estimation of Performance Measures in a Single M/Ek/1 Queue

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Abstract

An $Erlang - k$ (E_k) distributed random variable can be represented as the sum of k independent exponentially distributed random variables with the same means. In an $M/E_k/1$ queueing model service process is assumed to follow Erlang distribution. Other than its conventional uses in traffic flow, scheduling, facility design, and telecommunication, such queueing model is widely used in manufacturing systems and inventory management to investigate their operational performance. In this paper, the focus is on estimating measures of performance such as traffic intensity, and the average queue size in a single $M/E_k/1/\infty/\infty$ queueing model based on number of customers present in the queue at successive departure epochs. Both classical and Bayesian methods of estimation are used to obtain the estimates. A comprehensive simulation study starting with the transition probability matrix has been carried out along with the comparison of errors associated with the estimates.

Keywords and Phrases: $M/E_k/1/\infty$ queue, Queue length, Performance measures, Bayes estimator, Root mean square error, Transition Matrix
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1 Introduction

Development of queueing theory imparts models that predict behaviour of systems attempt to provide service for randomly arriving demands. Queues (or waiting lines) help facilities or systems to provide service in an orderly fashion. Any conclusion about a queueing problem comes from analyzing the model representing the queue. Because of the stochastic nature of queueing systems, the analysis is based on building a mathematical model representing a process of arrival of customers who join the queue, the rules by which they are allowed into service and the time it takes to serve the customers There exists a plethora of queueing models to represent the wide variety of service systems.The most common queueing models assume that inter-arrival time and service time follow exponential distribution or, equivalently, that number of arrivals and number of service completions follow Poisson distribution with one server having infinite capacity where customers arrive on first come first served basis ($FCFS$ queue discipline) from an infinite population size and it is denoted by $M/M/1/FCFS/\infty/\infty$. In one type of non-Markovian models, arrival or service process is assumed to follow Erlang distribution, e.g. models denoted by $M/E_k/1$ and $E_k/M/1$. Other queueing models of interest are $M/G/1$, $G/M/1$, $GI/G/1$ where inter-arrival time and service time have general or unspecified distributions (though with known means and variances). Other than its conventional uses in traffic flow, scheduling, facility design, and telecommunication, such queueing models are widely used in many other real-life situations to investigate their operational performance e.g. in manufacturing systems (Govil and Fu, 1999; Koole and Mandelbaum, 2002), in inventory management (Gayon et al. 2009, Ohta et al 2017), in health care (Almehdawe et al., 2013, 2016), in insurance (Ramirez et al 2010) and in other related areas.

Many performance measures of queueing systems are important indicators of their productivity and also the critical dimensions of the service quality. These performance measures are often quantitatively estimated using corresponding queueing performance metrics (QPMs), such as average queue length (L_q), average system length (L_s), average waiting time in system or mean sojourn time (W_s), and waiting time in queue (W_q) (cf. Chowdhury and Mukherjee, 2016). Generally, complete specified distributions are used for the input variables in a queueing model namely inter-arrival time and service time. Consequently, distributions of output variables like number present in the system (system size), waiting time etc. are derived in terms of the given distributions of input variables. Another important effectiveness measure is traffic intensity ($\rho = \frac{\lambda}{\mu}$) which is a measure of average occupancy of a server or in other words probability of the server being busy during a specified duration of time. In real life, the assumption of a particular form for the distribution of an input variable may be justified from prior considerations, but numerical values of input parameters (λ) and (μ) of these distributions are not given to us. Although traffic intensity is an effectiveness measure, it can also be treated as an input parameter in different queueing models, especially in $M/M/1$ and $M/E_k/1$ models. So, estimation of these input parameters as well as corresponding QPMs which are non random functions of these input parameters is essential for a better decision making.

Estimation of different performance measures are carried out by different researchers using both the maximum likelihood (ML) principle and the Bayesian method. A key step in the estimation techniques in the queuing theory involves the specification of likelihood function based on output measures. Such measures can be observed under three different setups:

(i) *Across independent queues* - Under this set-up, observations are taken on one or more relevant random variables from a number of identical queuing systems which form iid samples and are used to estimate both ML and Bayes estimates of QPM's. Aigner (1974) used time between successive arrivals, number

of items in a queue, system waiting time and service time across n iid $M/M/1$ queues to obtain ML estimators of queuing parameters. While Mukherjee and Chowdhury (2005, 2010) and Almeida and Cruz (2017) used queue length data across n iid $M/M/1$ queues to obtain ML and Bayes estimates of traffic intensity and related output measures, Cruz et al. (2017) used the same data in $M/M/s$ queueing systems to obtain the same set of outputs. Moreover, Chowdhury and Mukherjee (2011) used waiting time data to obtain ML and Bayes estimates of arrival and service rate parameters along with exceedance probabilities.
(ii) *A single queue with one or more independent random variables* - This situation was addressed by many researchers in both Markovian and non Markovian systems where data was taken from one or more independent variables from a single queue. Moran (1951) first obtained MLE of birth and death rates in a simple birth and death process. The landmark paper under this set-up was due to Clarke (1957) where the queue was observed for a length of time t and the number of arrivals, number of service completions and the time spent in the empty state were considered to yield iid sample. Later, MLE of input parameters and output measures are obtained by Wolff (1965), Cox (1965), Samaan and Tracy (1981), Basawa and Prabhu (1981), Basawa et al. (1996), Almeida et al. (2017) in the context of single $M/M/1$ and $M/M/s$ queueing systems observing independent random variables like number of arrivals, number of service completion, initial queue length, inter arrival times, and service times. Under the same set-up, Bayes estimates are obtained by Muddapur (1972), Armero (1985, 1994), McGrath and Singpurwalla (1987), Thiruvaiyaru and Basawa (1992), Armero and Bayarri (1994), Choudhury and Borthakur (2008), Almeida et al. (2017), Quinino and Cruz (2017). Number of arrivals during successive service periods was observed to obtain ML and Bayes estimators of traffic intensity in $M/E_k/1$ system by Harishchandra and Rao (1988) and Chowdhury and Maiti (2014). Bayes estimates of arrival and service rates are obtained in $M/E_k/1$ system by Wiper (1998) and Insua et al. (1998) and related hypothesis are tested in Bhat et al. (1997) is an overview of Bayesian and ML estimation.
(iii) *Single queue with dependent observations* - It deals with a single queue observed over a fixed length of time, where the random variable values are not mutually independent. Chowdhury and Mukherjee (2013) obtained estimators in a single $M/M/1$ queue under this set-up. Embedded Markov chains facilitate the study of this type of stochastic process in a queuing system such as M/G/1 as a Markov chain (see Bhat and Bhasawa, 2002; Basawa et al., 1996).

We use the third set-up in our study. In Section 2 of this article, likelihood function is developed on the basis of queue length at each departure epoch and MLE of performance measures are obtained in an $M/E_k/1/FIFO/\infty$ queuing system in equilibrium. The Bayes estimators are obtained in section 3 using squared error loss function (SELF) and precautionary error loss function (PLF). Beta and truncated uniform distributions are used as prior distributions to obtain the Bayes estimators. Section 4 presents and discusses the computational results obtained through simulating transition probability matrices. Section 5 concludes the paper.
In this article, we make use of a special function namely hyper-geometric function (available at R in 'hypergeo' package) denoted by $2F_1$ [a, b, c, z] which has the integral form

$$\frac{1}{\beta(b,c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}dt \tag{1.1}$$

2 Methodology

2.1 Single $M/E_k/1$ queue and observed data

An $Erlang - k$ (E_k) distributed random variable can be represented as the sum of k independent exponentially distributed random variables with the same means. In practical situations where observed data might not bear out the exponential distribution assumption, the Erlang can provide better flexibility by being better able to represent the real world (cf. Gross and Harris, 1998). In an $M/E_k/1$ queueing system, while the arrival process (inter-arrival time) is assumed to follow exponential distribution with pdf

$$a(t) = \lambda e^{-\lambda t} \; ; \; t > 0,$$

the service process (service time) follows Erlang distribution with pdf

$$b(t) = \frac{\mu k(\mu k t)^{k-1}}{(k-1)!} e^{(-\mu k t)} \; ; \; t > 0,$$

where $\frac{1}{\lambda}$ and $\frac{1}{\mu}$ are the mean arrival and service rates respectively with traffic intensity $\rho = \frac{\lambda}{\mu}$ assuming that the queue is observed for sufficiently long time to reach a steady state. Mean queue length which is a function of ρ is also given as $L_q = \frac{(r+1)\rho^2}{2r(1-\rho)}$

The queue length of an $M/E_k/1/FIFO/\infty$ queuing system is observed at successive departure epochs under steady state. Suppose N_t be the number of customers in the system immediately after t^{th} departure. The process N_t; t = 0, 1, 2, ... is a Markov Chain and the observations are dependent on each other.

Let A_t be the number of customers arriving during the service of t^{th} customer. The distribution of A_t is as follows

$$\begin{aligned} P(A_t = x) &= \int_0^\infty \frac{e^{-\lambda t}(\lambda t)^x \; \mu k(\mu k t)^{k-1} e^{(-\mu k t)}}{x! \, \Gamma(x)} \; ; \; t > 0, \; k \in 0, 1, ...\infty \\ &= \binom{x+k-1}{k-1} \left(\frac{\rho}{\rho+k}\right)^x \left(\frac{k}{\rho+k}\right)^k \; ; \; k \in 0, 1, ...\infty \qquad (2.2) \end{aligned}$$

Let P_{ij} be the transition probability $P(N_{t+1} = j | N_t = i)$ where N_t is the number of customers in the system immediately after t^{th} departure i.e the probability that the system will have j customers right after t^{th} departure when it has i customers.

$$\begin{aligned} N_{t+1} &= N_t - 1 + A_{t+1} \qquad if \; N_t > 0 \\ &= A_{t+1} \qquad if \; N_t = 0 \end{aligned}$$

Hence the conditional probabilities of N_t are given as

$$\begin{aligned} P(N_{t+1} = j | N_t = i) &= P(A_{t+1} = j) \qquad if \; i = 0 \\ &= P(A_{t+1} = j - i + 1) \qquad if \; i > 0 \end{aligned}$$

Hence, for an M/E_k/1 system, the transition probabilities are given by

$$
\begin{aligned}
P_{ij} &= \binom{j+k-1}{k-1}\left(\frac{\rho}{\rho+k}\right)^{j}\left(\frac{k}{\rho+k}\right)^{k} \;\; ; \; i=0,\; j=0,1,...\infty \\
&= \binom{j+k-i}{k-1}\left(\frac{\rho}{\rho+k}\right)^{j-i+1}\left(\frac{k}{\rho+k}\right)^{k} \;\; ; \; i\geq 1,\; j=(i-1)...\infty
\end{aligned}
\tag{2.3}
$$

2.2 Maximum Likelihood Method of Estimation

Suppose the process is observed till the number of departures reaches a particular value n. A single M/E_k/1 queue is observed and data on the system size is noted at successive departure epochs. If N_t be the number of customers present in the system immediately after t^{th} departure, then the process N_t; t = 0, 1, 2,...∞ is a Markov chain.

Hence, for the random variables $(N_0, N_1, ..., N_n)$ the likelihood function can be expressed in terms of transition probabilities given by

$$L(\rho) = \prod_{t=1}^{n} P(N_t = j | N_t = i)$$

It becomes convenient to track the state pairs (i,j), such that the queue length at consecutive departure epochs rather than actual number N_t. Some of the pairs may not be admissible while several others may be observed quite a few times. Hence, it is reasonable to consider the observed number of transitions for each state pair as our data. Let n_{ij} be the observed number of transitions in N_t from state i to state j such that the total number of transitions is N. Then, the likelihood function becomes

$$L(\rho) = \prod_{i=0}^{\infty}\prod_{j=i-1}^{\infty} P_{ij}^{n_{ij}} \tag{2.4}$$

Hence the Log-likelihood is given by

$$
\begin{aligned}
\ln L(\rho) &= \sum_{j=0}^{\infty} n_{0j}\ln p_{0j} + \sum_{j=0}^{\infty} n_{1j}\ln p_{1j} + \sum_{i=2}^{\infty}\sum_{j=i-1}^{\infty} n_{ij}\ln p_{ij} \\
&= \sum_{j=0}^{\infty}(n_{0j}+n_{1j})(constant + j\ln\rho - (j+k)\ln(k+\rho)) \\
&+ \sum_{i=2}^{\infty}\sum_{j=i-1}^{\infty} n_{ij}(const. + (j-i+1)\ln\rho - (j+k-i+1)\ln(k+\rho))
\end{aligned}
\tag{2.5}
$$

Upon solving, we get

$$\hat{\rho_{mle}} = \frac{\sum_{j=0}^{\infty} j(n_{0j}+n_{1j}) + \sum_{i=2}^{\infty}\sum_{j=i-1}^{\infty}(j-i+1)n_{ij}}{N_{00}+N_{10}+N} \tag{2.6}$$

where $N = \sum_{i=2}^{\infty}\sum_{j=i-1}^{\infty} n_{ij}$; $N_{00} = \sum_{j=0}^{\infty} n_{0j}$; $N_{10} = \sum_{j=0}^{\infty} n_{1j}$

The ML Estimator is independent of shape parameter k of the Erlang distribution i.e MLE method does not take into account the k different stages of a job and theorizes that it is one single job. This makes the Bayesian estimation essential.

2.3 Bayesian Method of Estimation

In the Bayesian framework, distribution of the the random variable X is not completely specified but depends on the unknown value θ of some parameter(s) with parameter space Θ, so that for each value of θ, X is distributed according to $p(x|\theta)$. Hence X is assumed to be a random sample, $X_1, X_2, ..., X_n$ so that $p(x|\theta) = \prod p(x_i|\theta)$. From a Bayesian point of view θ is considered as a random variable. Before X is observed, the a priori information about θ is quantified by prior distribution $p(\theta)$. After X is obtained, the posterior distribution of θ is given by

$$p(\theta|x) = \frac{p(x|\theta)p(\theta)}{p(x)}; \theta \in \Theta, \tag{2.7}$$

where $p(x) = \int p(x|\theta)p(\theta)d\theta$ is constant for given x and $p(x|\theta)$ is the likelihood function $L(\theta)$ of θ. Hence $p(\theta|x)$ is most commonly computed as

$$p(\theta|x) \propto L(\theta)p(\theta); \theta \in \Theta, \tag{2.8}$$

where the proportionality constant makes $p(\theta|x)$ integrate to one.
In this section Bayes estimators of ρ and L_q are obtained under squared error loss function (SELF) and precautionary error loss function (PLF). Beta distribution is used as prior distributions to obtain the Bayes estimators.

2.3.1 Bayes estimator under SELF

Beta distribution with hyper-parameters $(\theta 1, \theta 2)$ is taken as the natural conjugate prior density for ρ, it being a pretty general distribution for a proportion that takes up various particular forms of interest to us. Using the following prior distribution

$$p_1(\rho|\ \theta_1,\ \theta_2) = \frac{\rho^{\theta 1-1}(1-\rho)^{\theta 2-1}}{\beta(\theta_1, \theta_2)}\ , \qquad 0\ <\ \rho\ <\ 1 \tag{2.9}$$

and the kernel of the likelihood shown in (2.4), the posterior density of ρ becomes

$$q(\rho|\ n_{ij}\ \theta_1,\ \theta_2) = k_1 \rho^{\theta_1 - 1}(1-\rho)^{\theta_2 - 1} \prod_{i=0}^{\infty} \prod_{j=i-1}^{\infty} P_{ij}^{n_{ij}}, \tag{2.10}$$

where the normalizing constant k_1 is such that

$$k_1 \int_0^1 q(\rho|\ n_{ij}\ \theta_1,\ \theta_2) = 1, \tag{2.11}$$

which gives

$$\frac{1}{k_1} = \int_0^1 \left[\frac{\rho^{\phi_1}(1-\rho)^{\theta_2 - 1}}{(k+\rho)_2^{\phi}}\right] d\rho \tag{2.12}$$

$$= k^{-\phi_2}\beta(\phi_1 + 1, \theta_2)_2F_1[\phi_2, \phi_1 + 1, \phi_1 + \theta_2 + 1; -1/k] \tag{2.13}$$

where

$$\phi_1 \quad = \quad \theta_1 + \sum_{j=0}^{\infty} j(n_{0j} + n_{1j}) + \sum_{i=2}^{\infty} \sum_{j=i-1}^{\infty} (j-i+1)n_{ij} - 1 \tag{2.14}$$

$$-\phi_2 \quad = \quad \sum_{j=0}^{\infty} j(n_{0j} + n_{1j}) + k\sum_{j=0}^{\infty}(n_{0j} + n_{1j}) + \sum_{i=2}^{\infty} \sum_{j=i-1}^{\infty} (j-i+1+k)n_{ij} \tag{2.15}$$

The Bayes estimator of ρ under SLF is given by

$$\hat{\rho}^B_{SELF} = E(\rho|\ n_{ij}\ \theta_1,\ \theta_2) \tag{2.16}$$

$$= \frac{\phi_1 + 1}{\phi_1 + \theta_2 + 1} \frac{{}_2F_1[\phi_2, \phi_1 + 2, \phi_1 + \theta_2 + 2; -1/k]}{{}_2F_1[\phi_2, \phi_1 + 1, \phi_1 + \theta_2 + 1; -1/k]}. \tag{2.17}$$

2.3.2 Bayes Estimator under PLF

Most of the Bayes procedures have been developed under the usual SE loss function, which is symmetrical and give equal importance to the losses due to overestimation and underestimation of equal magnitude.There are situations where an underestimate is more serious than overestimate. In this case, use of symmetrical loss function might be inappropriate and a useful asymmetric loss function (Precautionary loss function) could be appropriate see Norstrom (1996). This loss function is interesting in the sense that a slight modification of squared error loss introduces asymmetry. The Precautionary loss function (PLF) is given by,

$$L_2(\hat{\theta}^B,\ \theta) = \frac{(\hat{\theta}^B - \theta)^2}{\hat{\theta}^B}, \tag{2.18}$$

where θ and $\hat{\theta}^B$ are parameter or parametric function and estimator respectively.
Minimizing $E_{\theta|\tilde{x}}L(\hat{\theta}^B,\ \theta)$, i.e. solving $\frac{dE_{\theta|\tilde{x}}L(\hat{\theta}^B,\ \theta)}{d\theta} = 0$, we get

$$\hat{\theta}^B = [E_{\theta|x}(\theta^2)]^{\frac{1}{2}} \tag{2.19}$$

Following the results as in (2.9)-(2.13), the Bayes estimator of ρ under PLF may be obtained as

$$\hat{\rho}^B_{PLF} \frac{(\phi_1 + 2)(\phi_1 + 1)}{\phi_1 + \theta_2 + 2} \frac{{}_2F_1[\phi_2, \phi_1 + 3, \phi_1 + \theta_2 + 3; -1/k]}{{}_2F_1[\phi_2, \phi_1 + 1, \phi_1 + \theta_2 + 1; -1/k]}. \tag{2.20}$$

3 Simulation Study

3.1 Procedure

To compare the various estimates of ρ and corresponding L_q for a fixed value of k, large simulation study has been performed. Different steps for the study are given below.

1. Total number of states (i, j) has been fixed at 15, i.e. $i = j = 0, 1, 2...14$.

2. Given $\rho = 0.5, 0.7, 0.9$, and i, generate random sample (j) of sizes 30, 50, 100 from transition probability distribution as shown in (2.3) using the *rnbinom* function in the software *R*. The i, j's are the number of customers present in the system at departure points and are states of the underlying Markov chain. Thus, eventual number of transitions constituting our data would be $N = 450, 750$.

3. Number of repetitions of (i, j) observed are recorded and an observed transition matrix is formed. If an observed transition state j corresponds to beyond the limits of j (i.e 14), j is assumed to be 14 (the upper bound). Six transition matrices are generated for these choices and are shown in the next subsection.

4. Using the derivations in Section 2, Bayes and ML estimates of ρ and L_q are computed from the transition matrix.

5. These steps are repeated 5000 times, hence estimates are generated, and their averages are taken. Average of root mean square error (rmse) is also considered as measure of sampling fluctuations. Here, $RMSE = \sqrt{\sum (T-\theta)^2/5000}$.

3.2 Results

Using the transition matrices 1-6, ML and Bayes estimates of the performance measures are obtained and are shown in Table 1. Following results are obtained from Table 1.

1. Bayes estimates perform better than the ML estimates in terms of RMSE. upon increasing the shape parameter k. However, the improvement in estimation decreases with increasing k.
2. Bayes estimator of ρ performs better under PLF than SELF. However, in most cases Bayes estimator of mean queue length performs better under SELF.
3. RMSE of all the estimates decrease with increase in the sample size.
4. Bayes estimates perform better at estimating parameters in terms of RMSE and practical implementation.
5. Erlangian service time model performs better than Markovian service time model in terms of RMSE.

4 Conclusion

Inferential methods based on a Bayesian approach were used to estimate the traffic intensity ρ for M/$_k$/1 queuing models. Using two forms of prior distributions, a beta distribution and a truncated uniform distribution, Monte-Carlo simulations indicated that the posterior estimates are very close to the known parameters. The root mean squared error was used to test for the most suitable prior distribution for the data. The extensive simulation study concludes that Erlangian service time model performs better than Markovian service time model.

Transition Matrix: 1

$\rho = 0.5,\ N = 450$

18	10	2	0	0	0	0	0	0	0	0	0	0	0	0
19	8	3	0	0	0	0	0	0	0	0	0	0	0	0
0	20	7	3	0	0	0	0	0	0	0	0	0	0	0
0	0	19	9	0	2	0	0	0	0	0	0	0	0	0
0	0	0	19	9	2	0	0	0	0	0	0	0	0	0
0	0	0	0	15	12	3	0	0	0	0	0	0	0	0
0	0	0	0	0	18	6	4	2	0	0	0	0	0	0
0	0	0	0	0	0	13	14	3	0	0	0	0	0	0
0	0	0	0	0	0	0	21	5	4	0	0	0	0	0
0	0	0	0	0	0	0	0	19	9	2	0	0	0	0
0	0	0	0	0	0	0	0	0	20	9	1	0	0	0
0	0	0	0	0	0	0	0	0	0	21	7	2	0	0
0	0	0	0	0	0	0	0	0	0	0	18	6	5	1
0	0	0	0	0	0	0	0	0	0	0	0	21	5	4
0	0	0	0	0	0	0	0	0	0	0	0	0	19	11

Transition Matrix: 2

$\rho = 0.5,\ N = 750$

33	14	2	0	0	1	0	0	0	0	0	0	0	0	0
31	16	3	0	0	0	0	0	0	0	0	0	0	0	0
0	27	17	3	2	1	0	0	0	0	0	0	0	0	0
0	0	34	12	3	1	0	0	0	0	0	0	0	0	0
0	0	0	35	13	2	0	0	0	0	0	0	0	0	0
0	0	0	0	38	9	2	1	0	0	0	0	0	0	0
0	0	0	0	0	31	17	2	0	0	0	0	0	0	0
0	0	0	0	0	0	34	15	1	0	0	0	0	0	0
0	0	0	0	0	0	0	28	21	0	1	0	0	0	0
0	0	0	0	0	0	0	0	30	14	3	3	0	0	0
0	0	0	0	0	0	0	0	0	36	9	5	0	0	0
0	0	0	0	0	0	0	0	0	0	27	20	2	1	0
0	0	0	0	0	0	0	0	0	0	0	28	16	5	1
0	0	0	0	0	0	0	0	0	0	0	0	36	11	3
0	0	0	0	0	0	0	0	0	0	0	0	0	31	19

Transition Matrix: 3

$\rho = 0.7,\ N = 450$

18	8	2	2	0	0	0	0	0	0	0	0	0	0	0
15	8	5	2	0	0	0	0	0	0	0	0	0	0	0
0	10	12	7	0	1	0	0	0	0	0	0	0	0	0
0	0	23	6	1	0	0	0	0	0	0	0	0	0	0
0	0	0	14	11	2	2	1	0	0	0	0	0	0	0
0	0	0	0	12	14	4	0	0	0	0	0	0	0	0
0	0	0	0	0	16	11	2	1	0	0	0	0	0	0
0	0	0	0	0	0	17	9	4	0	0	0	0	0	0
0	0	0	0	0	0	0	15	10	3	2	0	0	0	0
0	0	0	0	0	0	0	0	15	10	4	0	1	0	0
0	0	0	0	0	0	0	0	0	18	4	6	2	0	0
0	0	0	0	0	0	0	0	0	0	17	11	2	0	0
0	0	0	0	0	0	0	0	0	0	0	16	5	7	2
0	0	0	0	0	0	0	0	0	0	0	0	18	10	2
0	0	0	0	0	0	0	0	0	0	0	0	0	18	12

Transition Matrix: 4

$\rho = 0.7,\ N = 750$

31	10	9	0	0	0	0	0	0	0	0	0	0	0	0
26	14	9	1	0	0	0	0	0	0	0	0	0	0	0
0	26	20	4	0	0	0	0	0	0	0	0	0	0	0
0	0	27	13	6	3	1	0	0	0	0	0	0	0	0
0	0	0	30	12	6	2	0	0	0	0	0	0	0	0
0	0	0	0	31	9	4	2	3	1	0	0	0	0	0
0	0	0	0	0	29	10	8	2	1	0	0	0	0	0
0	0	0	0	0	0	24	14	10	2	0	0	0	0	0
0	0	0	0	0	0	0	27	20	2	1	0	0	0	0
0	0	0	0	0	0	0	0	26	11	8	4	0	1	0
0	0	0	0	0	0	0	0	0	17	23	7	3	0	0
0	0	0	0	0	0	0	0	0	0	28	11	5	4	2
0	0	0	0	0	0	0	0	0	0	0	23	13	11	3
0	0	0	0	0	0	0	0	0	0	0	0	28	14	8
0	0	0	0	0	0	0	0	0	0	0	0	0	29	21

Transition Matrix: 5

$\rho = 0.9,\ N = 450$

16	10	3	1	0	0	0	0	0	0	0	0	0	0	0
15	12	2	1	0	0	0	0	0	0	0	0	0	0	0
0	13	13	3	0	1	0	0	0	0	0	0	0	0	0
0	0	14	10	4	2	0	0	0	0	0	0	0	0	0
0	0	0	11	14	4	1	0	0	0	0	0	0	0	0
0	0	0	0	9	12	7	2	0	0	0	0	0	0	0
0	0	0	0	0	12	9	9	0	0	0	0	0	0	0
0	0	0	0	0	0	10	13	3	4	0	0	0	0	0
0	0	0	0	0	0	0	14	9	5	1	1	0	0	0
0	0	0	0	0	0	0	0	14	12	3	1	0	0	0
0	0	0	0	0	0	0	0	0	14	7	4	1	3	1
0	0	0	0	0	0	0	0	0	0	14	10	4	2	0
0	0	0	0	0	0	0	0	0	0	0	17	8	1	4
0	0	0	0	0	0	0	0	0	0	0	0	14	11	5
0	0	0	0	0	0	0	0	0	0	0	0	0	11	19

Transition Matrix: 6

$\rho = 0.9,\ N = 750$

26	13	7	3	1	0	0	0	0	0	0	0	0	0	0
25	13	9	2	0	0	1	0	0	0	0	0	0	0	0
0	17	21	9	1	1	1	0	0	0	0	0	0	0	0
0	0	20	17	9	3	0	0	1	0	0	0	0	0	0
0	0	0	21	22	6	0	1	0	0	0	0	0	0	0
0	0	0	0	22	21	5	1	0	1	0	0	0	0	0
0	0	0	0	0	22	16	7	2	2	1	0	0	0	0
0	0	0	0	0	0	20	19	7	2	1	0	1	0	0
0	0	0	0	0	0	0	30	11	6	3	0	0	0	0
0	0	0	0	0	0	0	0	16	21	9	4	0	0	0
0	0	0	0	0	0	0	0	0	18	16	14	2	0	0
0	0	0	0	0	0	0	0	0	0	18	18	7	4	3
0	0	0	0	0	0	0	0	0	0	0	19	21	7	3
0	0	0	0	0	0	0	0	0	0	0	0	24	16	10
0	0	0	0	0	0	0	0	0	0	0	0	0	21	29

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Table 1: ML and Bayes estimates under SELF and PLF

	n	$\hat{\rho}_{MLE}$	$\hat{\rho}_{SELF}$	$\hat{\rho}_{PLF}$	L_{qSELF}	L_{qPLF}
$k = 1, \rho = 0.5$	50	0.4534	0.5210	0.5128	0.9794	3.9740
$L_q = 0.5$		(0.1139)	(0.0961)	(0.0907)	(0.9280)	(5.1099)
	100	0.4754	0.5174	0.5112	0.7065	1.3324
		(0.0841)	(0.0799)	(0.0701)	(0.4824)	(1.7103)
$k = 4, \rho = 0.5$	50	0.4699	0.5076	0.5227	0.9604	1.4467
$L_q = 0.312$		(0.0990)	(0.0947)	(0.0919)	(0.8712)	(2.0090)
	100	0.4864	0.5084	0.5064	0.7067	0.4014
		(0.08491)	(0.0784)	(0.0597)	(0.4829)	(0.1564)
$k = 1, \rho = 0.7$	50	0.6377	0.6394	0.6496	2.6355	11.2479
$L_q = 1.633$		(0.1121)	(0.1084)	(0.1043)	(2.106)	(12.2132)
	100	0.6518	0.6736	0.6771	2.6342	8.4227
		(0.1084)	(0.0874)	(0.0843)	(2.0083)	(10.0890)
$k = 4, \rho = 0.7$	50	0.6370	0.6598	0.6702	2.6351	6.3031
$L_q = 1.020$		(0.1273)	(0.0975)	(0.0801)	(2.5178)	(7.3857)
	100	0.6691	0.6792	0.6882	2.5695	4.2362
		(0.0906)	(0.0862)	(0.0772)	(2.1121)	(5.1215)

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